Walter Carnielli

The Single-minded Pursuit of Consistency and its Weakness

Saberíamos muito mais das complexidades da vida se
nos aplicássemos a estudar com afinco as suas contradições
em vez de perdermos tanto tempo com as identidades e coerências,
que estas têm obrigação de explicar-se por si mesmas.[1]
José Saramago, "A Caverna"

Abstract. I argue that a compulsive seeking for just one sense of consistency is hazardous to rationality, and that observing the subtle distinctions of reasonableness between individual and groups may suggest wider, structuralistic notions of consistency, even relevant to re-assessing Gödel's Second Incompleteness Theorem and to science as a whole.

Keywords: Consistency, contradictorieness, inconsistency non-standard models, structuralism.

1. Requirements for rationality

We are more or less used to the idea that public and private beliefs are distinct concepts, but why are we hesitating in sustaining that public and private knowledge coincide? Are public and private rationality just the same?

Criteria for rationality abound, but some perspicuous views are attempted in some papers in Mele [35]. The role of contradictory beliefs is particularly scrutinized in Rovane [39] while discussing the basis for rationality. Requirements for rationality, essentials for a person's deliberation, according to Rovane [39] (p. 322), are the resolutions of contradictions among one's beliefs, consequence (implications of one's beliefs in view of other attitudes) and ranking one's preferences in a transitive ordering. Thus, someone trapped on contradictory beliefs might be acting in the opposite way to the search of the best action, since contradictory beliefs would force antagonistic courses of action. But contradictions do not necessarily produce logical anarchy, as noted by many, particularly in Rescher and Brandon [37], section 6. To have at our disposal robust logics whose deductive machinery does not

[1]We would know more about life's complexities if we applied ourselves to the study of its contradictions instead of wasting so much time on identities and similarities, because these carry the obligation to be self-explanatory. Saramago [41] [My translation]

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collapse when facing a contradiction is not to cherish contradictions, but just to maintain the most rational attitude while deciding among disharmonious beliefs.

But there may be striking differences between normative requirements for individual rationality and for group rationality. Requirements like doxastic non-contradictoriness, so maintains Rovane [39]:

> ... apply only to individual persons and not to groups of them. This can be seen from our critical reactions. If one person believes two contradictory propositions, then we are bound to regard this as a failure of rationality on that person's part. But if one person holds one belief while another person believes its contrary, then we are not bound to regard either person as guilty of a rational failure. For each of them might have reasoned correctly from its own point of view, by arriving at all-things-considered judgments that take all of its background attitudes into account.

The intuition behind this consideration fits surprisingly well with the motivations for the *society semantics* and for the *possible-translations semantics*, new kinds of semantics pertinent to several contemporary logics, to be explained in more details in the next section. The point is that agents who disagree among themselves are not necessarily under "rational pressure to resolve their disagreements" (cf. Rovane [39]), while self-disagreement may be more demanding. A different line of argumentation, but going in the same direction, is developed in Massaci [34].

2. Possible-translations semantics and society semantics

I have argued elsewhere that rationally acceptable statements are not coincident with true statements, and contradictions in reasoning sometimes play a most important role. So, for instance, only contradictory allegations can make a judge decide whether there is any false statement around: indeed, there is no other reason why, in justice, people are interviewed in separate. Much to the contrary to what some philosophers maintain, contradictions are precious: in informal reasoning the use of contradictions is inherent, and it is a challenge for logic to offer a suitable formal model for the perfectly licit act of reasoning under contradictions, and paraconsistent logic strives to meet this challenge. In particular, the wide family of logics of formal inconsistency (LFI's) (cf. Carnielli, Coniglio and Marcos [15]) achieves this in a remarkably natural and elegant way, but new proof methods, and specially new semantics are required. The possible-translations semantics (PTS's) were devised in 1990 (cf. Carnielli [11]) in order to offer a palatable semantical interpretation for some non-classical logics, and are very well adapted to paraconsistent logics, in particular to LFI's. Improved recent expositions, correcting some mistakes and extending the idea, appear in Marcos [30] and in Carnielli [12]; A well-organized and systematic search for PTS's applied to several logic systems is done in Marcos [33]. The notion of *translations* as morphisms between logics (maps preserving their consequence relations) is essential in the PTS's. Translations (for a recent treatment on this topic see Carnielli, Coniglio and D'Ottaviano [13]) can be thought as different "world views", and the concept of possible-translations semantics offer a way to interpret a given logic L as a sheaf bounding all possible world views. Technically each translation provides a world view by mapping formulas of L into a class of "simpler" logics $L_1, \ldots, L_n, \ldots$ with known semantic characterization. In this way LFI's give a plain natural account of understanding the phenomenon of a sentence and its negation as being both true. But PTS's can be given to non-paraconsistent logics as well.

Many paraconsistent logics which are not characterizable by means of finite matrices can be characterized by suitable combinations of many-valued logics. PTS's also serve as a powerful tool to decompose logics in the spirit of the program of splitting and splicing logics (cf. Carnielli *et allia* [14]). It is interesting to remark that PTS's can even give a solution to the sneaky problem of algebrizing a paraconsistent logic (see Bueno-Soler, Carnielli and Coniglio [8] for a full explanation of the problem and for further references).

The possible-translations semantics and the society semantics act in good correspondence with the above mentioned (Rovane [39]) rationality requirements for individuals (or agents) and for groups of them. Actually, such semantics show clearly that the internal logic of a group of individuals may be strikingly different from the logic of its members. In other words, society semantics (better applied to many-valued, truth-functional situations) are able to establish, in general, that certain n-valued semantics can be expressed in terms of groups of m-valued semantics, where $m \leq n$. For instance, in Carnielli and Lima-Marques [16] it is shown that the logic of a society acting under certain rules and formed by just a couple of agents equipped with propositional classical logic coincides with the three-valued paraconsistent logic $\mathbf{P^1}$. This happens, of course, if the members of this society "agree on disagree", that is, if their differences are regulated by certain conditions which are said to constitute an *open society*. They could also regulate their society differently (by means of a *closed society*), in which case the logic of their society coincides with the three-valued paracomplete (or weakly-intuitionistic) logic $\mathbf{I^1}$.

The possible-translations semantics, introduced earlier (of which society semantics are a heir), and with a more ambitious scope, can be applied to several situations where the logics are not even truth-functional. Possible-translations semantics constitute a more general setting since translations are involved, and even many-valued logics (as the well-known Łukasiewicz's L_3 system) can be given a possible-translations semantics, besides a society semantics (in this case, by translating L_3 sentences into two-valued logics, cf. Marcos [30]; further treatment in Caleiro *et allia* [10]).

An important feature of possible-translations semantics, which is widely used e.g. in Carnielli, Coniglio and Marcos [15] to provide such semantics to LFI's, is to split the consistency operator $\circ$ and the inconsistency operator $\bullet$ in terms of many-valued factors, by means of appropriate translations. This gives a nice operational meaning to consistency and inconsistency operators, but questions about the significance of these notions of consistency and inconsistency still stand: what is their meaning? Is consistency an absolute concept?

3. Is consistency consistent?

A critical view on "what is consistency?" seems to be a crucial matter, though largely overlooked by logicians and philosophers. The underlying intuition of consistency varies from harmonious, consonant or coherent, unchanging, constant or continuous, compatible and concordant. It is not difficult, indeed, to find sentences in natural language conveying all such meanings.

In some of the above accounts, consistency has nothing to do with negations. However, in the celebrated Gödel's theorems consistency is assumed as a syntactical notion: the consistency operator $Consis(A)$ means that "A is consistent with Peano Arithmetic (**PA**)" in a way connected to negation. This idea of consistency is precisely "contradiction-free" (in **PA**, or in any formal system or theory **T**, in the sense that there is no sentence B such that $A \vdash_{\mathbf{T}} B$ and $A \vdash_{\mathbf{T}} \neg B$). By using the device of Gödel numbering, one can (as it is well known) produce an unprovable sentence G and establish (by a finitary proof) that G is true (and unprovable) if and only if **T** is consistent, for any **T** which contains as much number theory as **PA**. It follows from this that **T** cannot prove its own consistency, if **T** is consistent.

But is this a universal conception of consistency? Is this the same idea that other people in other areas profess about consistency?

Apparently, Gödel never explicitly admitted, in regard to his Second Incompleteness Theorem, that it should be impossible for a generic system with some amount of arithmetic to prove its own consistency by some kind of finite means. For instance, it can be shown (cf. Willard [48]) that first-order systems of arithmetic that self-verify their own consistency are possible, if we weaken their arithmetical capacity so that they could not formalize diagonalization. This can be done e.g. if multiplication is not a total function anymore. If subtraction and division are primitive function symbols in a weak version of first-order arithmetic, addition and multiplication can be defined upon them in such a way that a sentence expressing totality of multiplication cannot be proven. As a consequence, the Gödelian diagonalization argument fails to apply, and not only the Second Incompleteness Theorem collapses, but the theories can prove its own consistency.

On the other end of the rainbow, it is possible to strengthen theories of arithmetic so as to make them inconsistent, while taming the underlying logic. As a result, again the Second Incompleteness Theorem fail, since such theories may contain undecidable sentences as well as their negations, as shown in Mortensen [36].

How can we make sense of such digressions around an idea of consistency that would seem to be so solid?

3.1. Consistency and belief revision

It does not seem to be so in various other areas. Alchourrón, Gärdenfors and Makinson have put forward (e.g., in [1]) a number of postulates that belief revision should satisfy, among them (for Γ a belief set, and $\Gamma * A$ the process of revising Γ taking A into consideration):

1) (Success) $A \in \Gamma * A$;
2) (Consistency) If both Γ and A, when considered separately, are logically consistent then $\Gamma * A$ is logically consistent;
3) (Preservation) If A is logically consistent with Γ, then $\Gamma \subset \Gamma * A$.

This seemingly acceptable theory of belief revision would seem to be in complete agreement with Ramsey's test, a check for acceptance of conditionals proposed in Stalnaker [46]:

> This is how to evaluate a conditional: First, add the antecedent (hypothetically) to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent); finally, consider whether or not the consequent is then true.

However, when taken together, the postulates of (Success), (Consistency) and (Preservation) conflict with the Ramsey test, as shown in Gärdenfors [20]. As argued in Wassermann [47], for instance, some of the inherent problems may be due to the view that rationality is assumed to be related (or confined) to classical logic. But even if changing the logic is somehow setting rationality free of "classical" restraints, the postulates of (Consistency) and (Preservation) are too dependent of an concealed use of consistency. But what is wrong with the notion of consistency tacitly assumed in belief revision?

3.2. A "logic of inconsistency"

According to Rescher and Brandon [37], it is desirable to avoid inconsistencies in our own thought, but not necessarily in the *objects* of this thought. So, at page 4:

> The consistent theoretical scrutiny of inconsistent worlds is not only an attainable but perhaps a useful goal.

The first thing we notice here is that "consistency" as treated by Rescher and Brandon in [37] is nothing else than contradictoriness. And they dodge the effects of contradictoriness by stipulating non-standard worlds: basically, the so-called *schematic* worlds where the law of excluded middle (*tertium non datur*) fails (and so there may be theses P such that neither P nor $\neg P$ holds in a schematic world), and the *inconsistent* or *superimposed* worlds where the law of non-contradiction fails (and so there may be theses P such that P and $\neg P$ hold in a inconsistent world).

In this way, given worlds $w_1, \ldots, w_n$, the so-called ontologically undetermined world $w_1 \uplus \ldots \uplus w_n$ is a schematic world such that, for any proposition P, P holds in $w_1 \uplus \ldots \uplus w_n$ iff P holds in all worlds $w_1, \ldots, w_n$. In similar fashion, the ontologically overdetermined world $w_1 \Cap \ldots \Cap w_n$ is a superimposed world such that, for any proposition P, P holds in $w_1 \Cap \ldots \Cap w_n$ iff P holds in at least one of $w_1, \ldots, w_n$.

Although the idea is not philosophically uninteresting, it is rather poor from the logic-mathematical viewpoint. The operations $\Cap$ and $\uplus$, lack structure (they are not, for instance, supposed to be associative, which demands that $w_1 \Cap \ldots \Cap w_n$ requires a 2^n valued internal logic (actually, the n-folded cartesian product of classical propositional logic.[2]

3.3. Consistency and strict implication

The concept of *strict implication* was introduced by Lewis and Langford in [25] by means of the systems **S1** to **S5** with the intention to avoid the "para-

[2] This logic in the truth-values $\langle x_1, \ldots, x_n\rangle$ (for $x_i \in \{0, 1\}$) is defined as $\neg\langle x_1, \ldots, x_n\rangle = \langle\neg x_1, \ldots, \neg x_n\rangle$, $\langle x_1, \ldots, x_n\rangle \vee \langle y_1, \ldots, y_n\rangle = \langle x_1 \vee y_1, \ldots, x_n \vee y_n\rangle$ and $\langle x_1, \ldots, x_n\rangle \wedge \langle y_1, \ldots, y_n\rangle = \langle x_1 \wedge y_1, \ldots, x_n \wedge y_n\rangle$. For $n = 2$ the same definitions have been used by Łukasiewicz in [28], and before that by Lewis in [26].

doxes of material implication" of *Principia Mathematica*, although some previous criticisms have been raised already by H. MacColl in [29]. By defining the strict implication $\prec$ by

$$p \prec q := \Box(p \supset q),$$

the non-normal system **S2** is given[3], following E. J. Lemmon's formulation as in Chapter 11 of Hughes and Cresswell [21], by the axioms and rules below:

1. every classical propositional (**PC**) tautology
2. $\Box p \supset p$
3. $\Box(p \supset q) \supset (\Box p \supset \Box q)$ (*Consistency Postulate*)
4. $\Diamond(p \wedge q) \prec (\Diamond p \wedge \Diamond q)$

Rules:

1. *Modus Ponens* for $\supset$
2. Restricted Necessitation Rule: if α is a **PC**-tautology or an axiom, then $\vdash \Box\alpha$
3. Becker's Rule: If $\vdash \Box(\alpha \supset \beta)$ then $\vdash \Box(\Box\alpha \supset \Box\beta)$

Some immediate properties are:

1. $(p \wedge q) \prec (q \wedge p)$
2. $(p \wedge q) \prec p$
3. $p \prec (p \wedge p)$
4. $((p \wedge q) \wedge r) \prec (p \wedge (q \wedge r))$
5. $((p \prec q) \wedge (q \prec r)) \prec (p \prec r)$
6. $(p \wedge (p \prec q)) \prec q$

Understanding $p \prec q$ as "q is deducible from p", then p is *consistent with* q is defined by[4]:

$$p \odot q := \Diamond(p \wedge q)$$

$p \odot q$ can be seen as a kind of "strong adjunction", and its more interesting properties are, defining $p \equiv q := (p \prec q) \wedge (q \prec p)$:

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[3] **S1** is a subsystem of **S2**, but details are not relevant here.

[4] Although the symbol $\circ$ is used in Lewis and Langford [25], I employ here $\odot$ to avoid confusion with the consistency connective of LFI's. Both notions are related, as we shall see, but not coincident.

- $p \odot q := \neg(p \prec \neg q)$
- $\Diamond p \equiv p \odot p$
- $(p \wedge q) \prec (p \odot q)$ (conjunction strongly implies consistency)
- $\neg(p \odot \neg p)$ (no proposition is consistent with its own denial)
- $p \odot q \equiv q \odot p$ (consistency is commutative)
- $p \prec p \odot p$ (a true proposition is self consistent)

However, it is interesting to remark that $\odot$ is not transitive (cf. Lewis and Langford [25], page 158), and the consistency of a finite set $p_1, \ldots, p_n$ means (cf. page 156) the joint assertion of any subset $p_1, \ldots, p_n$ is consistent with the joint assertion of the remainder of the set. Thus the consistency of p, q, r is neither $p \odot (q \odot r)$, nor $p \odot q(\odot r)$, but the equivalents $(p \wedge q) \odot r$ or $p \odot (q \wedge r)$.

Several interesting consequences follow:

- $p \odot q \prec p \odot p$ (a proposition consistent with any other is consistent with itself)
- $\neg\Diamond p \prec (p \prec q) \wedge (p \prec \neg q)$ (an impossible proposition strongly implies a contradictory pair)
- $(q \prec p) \wedge (\neg q \prec p) \prec \Box p$ (a contradictory pair strongly implies a necessary proposition)
- $\neg\Diamond\neg p \supset (q \prec p)$
- $\neg\Diamond p \supset (p \prec q)$

But there are also some important negative results: the last two laws do not obtain in strict form: $\neg\Diamond\neg p \prec (q \prec p)$ and $\neg\Diamond p \prec (p \prec q)$ are not valid, and no thesis of the form $\Box\Box\alpha$ can be obtained in **S2**. This one, in particular, means (for $\neg\alpha$) $\neg\Diamond\Diamond\alpha$, or, in plain words, that the impossibility of consistency cannot be proved.

The system **S3** is obtained from **S2** by adding a sentence related to Becker's Rule:

$$\Box(\alpha \supset \beta) \supset \Box(\Box\alpha \supset \Box\beta)$$

S4 and **S5** coincide with their familiar contemporary systems.

A pertinent point here is that the notion of consistency in **S2** can be made *totally independent* of negation. Indeed, consistency (and strict implication) can be defined within the absolutely positive modal logics as in Bueno-Soler [7], for $\supset$ the material implication:

$$p \prec q := \Box(p \supset q)$$

$$p \odot q := \Diamond(p \wedge q)$$

In this way, all we can know about consistency would reduce to what we can know about positive possibility.

S2 was considered by Lewis as giving the most adequate account of possibility and necessity, and it is well known that **S2** represented a challenge in terms of formal semantics: J. Dugundji had shown in Dugundji [19] that no systems between **S1** and **S5** could be semantically characterized by finite-valued truth-functional semantics, and only 15 years later Kripke in [23] devised a way to give a semantics to what he called non-normal modal logics.

The semantics is given by a *non-normal frame* $\langle W, N, R, v\rangle$ where W are worlds, R are relations and N are *non-normal worlds* and the valuation v is such that $v(\Box p) = F$ at any non-normal $w \in N$, while $v(\Diamond p) = V$ at any non-normal $w \in N$. So in non-normal worlds $\Diamond(p \wedge \neg p)$ is true, but it does not mean that $p \wedge \neg p$ is true, since a non-normal world is not related to any other world, not even to itself. Thus are non-normal worlds in which contradictions are merely possible, but not where contradictions are true.

This may be a very positive feature, as pointed out in Shukla [44], in the sense that any decent modal logic ought to have propositions that are true but not necessary (so as to distinguish contingently true from necessary), and its counterpart, propositions that are false but not impossible.

The concept of non-normal world, even if mathematically flawless and exhibiting the mentioned positive feature, is not devoid of afflictions: rephrasing Lewis in [26] (p. 183 and ff.), we could not say, without falling into circularity, that a consistent sentence is one that "could" be true (or, equivalently, one that is not necessarily false):

Possible worlds, if they exist, cannot be reduced to anything more basic, and other possible worlds are things of the same kind as the actual world, as Stalnaker summarizes D. Lewis' modal realism (a.k.a. possibilism) in Stalnaker [45]. This does not make easy to accept non-normal words as legitimate "possible worlds".

4. Modal approaches to consistency and inconsistency

The first idea of paraconsistent logic was modal: indeed, Jaśkowski's discussive system **D2** was outlined in Jaśkowski [22] 60 years ago as a system which could include disagreement between a society and its individuals. The statements of such a logic could tolerate contradictions, if preceded by some cautions (as ‘in accordance with the opinion of one of the participants in the discussion’, or ‘for a certain admissible meaning of the terms’, etc). **D2** is given by an implementation as a fragment of **S5**.

As shown in Marcos [31], however, **D2** is paraconsistent (indeed an LFI, see also Caleiro *et allia* [10]) but not itself a modal logic in the contemporary sense (albeit being a fragment of a modal system).

On the other direction, however, every normal (and non-degenerate[5]) modal logic contains an underlying paraconsistent logic, as defended in Marcos [31] and even before in Béziau [3] and [4]: a paraconsistent negation can be defined in any such normal modal logic which has negation $\neg$ and possibility $\Diamond$ as:

$$\sim \alpha := \Diamond\neg\alpha.$$

This is, in a certain sense, a dual of the fact that intuitionistic negation has a modal interpretation in terms of a translation into **S4**, interpreting intuitionistic negation by $\neg\Diamond$.

Moreover, a sort of consistency connective $\oslash$ (similar of $\circ$ in the LFI's) can also be defined in a normal non-degenerate modal logic (if $\Box$ is in the language, or defining as usual $\Box$ as $\neg\Diamond\neg$) by

$$\oslash\alpha := \alpha \supset \Box\alpha$$

or, equivalently, by $\oslash\alpha := \alpha \supset \neg \sim \alpha$.

It is also possible, following Marcos [31], to start from a purely positive “modal logic of consistency” having in the language the classical propositional connectives $\wedge$, $\vee$ and $\supset$, plus a consistency connective as primitive, interpreted in a Kripke-style relational model as:

$$\models_x^{\mathcal{M}} \oslash\alpha \;\text{ iff }\; \models_x^{\mathcal{M}} \alpha \text{ implies } (\forall y)(\text{if } x\mathsf{R}y \text{ then } \models_y^{\mathcal{M}} \alpha).$$

Now, if the “modal logic of consistency”, instead of positive, had a weak (paraconsistent) negation, a classical negation could be defined by setting the following definition (where $\circ$ is the consistency operator of LFI's):

$$\neg\alpha := \alpha \to (\alpha \wedge (\sim \alpha \wedge \circ\alpha)).$$

As it is well known, most elementary modal axioms are but special cases of the schema $G^{k,l,m,n}$, where numerical indices stand for the number of iterated operators:

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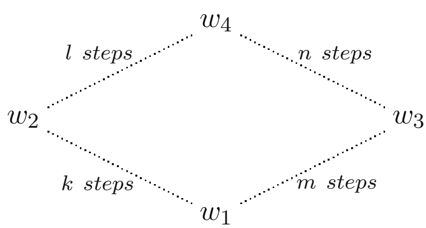
[5]A normal modal logic is degenerated if it is characterized by frames such that every world can access only itself or no other world.

$G^{k,l,m,n}$: $\Diamond^k \Box^l p \supset \Box^m \Diamond^n p$

Every extension of the basic modal system **K** by an instance of $G^{k,l,m,n}$ is characterized (i.e., sound and complete) with respect to the so-called confluential (or Church-Rosser) frames whose accessibility relations are defined in first-order as:

$\mathbf{C}^{k,l,m,n}$: $\forall w_1 \forall w_2 \forall w_3((w_1 R^k w_2 \wedge w_1 R^m w_3) \supset \exists w_4(w_2 R^l w_4 \wedge w_3 R^n w_4))$

This characterization works well for any finite number of such instances of schema $G^{k,l,m,n}$ (for detail, check Carnielli and Pizzi [18], chapter 4). The confluential frames can be visualized by the following graph:



Therefore, assuming $\oslash\alpha := \alpha \supset \Box\alpha$ as an axiom amounts to assuming $G^{0,0,1,0}$ added to **K**, which results in the modal logic of "global consistency"[6] characterized by frames with the property:

$\mathbf{C}^{0,0,1,0}$: $\forall w_1 \forall w_2(w_1 R w_2 \supset w_1 = w_2)$

It can be easily checked that this consistency connective $\oslash$, the defined negation $\sim$ and the connective $\otimes\alpha := \alpha \wedge \sim \alpha$ have all the relevant features of negation, consistency and inconsistency of the LFI's:

- $\alpha, \sim \alpha \nvdash \beta$ for an arbitrary β
- $\oslash\alpha, \sim \alpha, \alpha \vdash \beta$, for all β
- $\otimes\alpha \equiv \neg \oslash \alpha$
- $\otimes\alpha \vdash \alpha$ and $\otimes\alpha \vdash \sim \alpha$
- $\otimes\alpha \not\equiv \alpha \wedge \neg\alpha$

Weaker forms of (modal) consistency can be defined by:

$\oslash_{\Diamond^k} p :=$ $\Diamond^k p \supset \Box\Diamond^k p$ with the meaning that just possibilities of

[6]Called TV in Marcos [32].

the form $\Diamond^k$ are consistent; a logic of "$\Diamond^k$ global consistency can be immediately axiomatized by adding instances of $G^{k,l,m,n}$

$$G^{k,0,1,k}: \quad \Diamond^k p \supset \Box\Diamond^k p$$

and obviously characterized by $\mathbf{C}^{k,0,1,k}$ confluential frames.

It can be easily verified that $\oslash\alpha := \alpha \supset \Box\alpha$ is non-normal, in the sense of not distributing well with respect to implication; indeed, an essential axiom in normal systems which should be held by the operator $\oslash$ is:

$$\oslash(p \supset q) \supset (\oslash p \supset \oslash p)$$

is not valid, as can be easily checked by a two-worlds counter model: w, w', wRw', where $w \not\vDash p, w \vDash q, w' \not\vDash p, w' \not\vDash q$. Clearly, $w' \vDash p \supset q$, and therefore $w \vDash \Box(p \supset q)$. Since $w \vDash p \supset q$, $w \vDash \oslash(p \supset q)$.

It is clear that $w \vDash \oslash p$ and $w \not\vDash \oslash q$, hence $w \not\vDash (\oslash(p \supset q) \supset (\oslash p \supset \oslash p))$.

The above sense of consistency ($\oslash$) has an essential "possible-world" character. But, as shown in Bueno-Soler [7], many *cathodic* modal logics can be defined adjoining modal structures on top of LFI's. The main ones are **KmbC**, **KbC** and **KCi**, obtained by extending the LFI's **mbC**, **bC** and **Ci** with the machinery of the modal systems **K** plus instances of the schema $G^{k,l,m,n}$.

All such systems have a consistency connective $\circ$, semantically characterized by possible-translations semantics. But in all of them modal consistency as the above $\oslash$ can be defined. It happens that systems of this sort have *two levels* of consistency, semantically characterized by possible-translations semantics!

We should not forget that consistency is related to existence in mathematics; even if, as defended by Bernays in [2], existence for mathematical entities is not synonymous with consistency, a mathematician would hardly deny that inconsistency (in the usual mathematical sense) entails non-existence. How can we thus explain this inflation of consistency, and how this affects what we believe not to exist?

5. Are we aware of all senses of consistency?

There seems to be thus several ways to refer to consistency, and to several attitudes towards consistency. We may distinguish, for instance, Aristotelian consistency as related to the notions of *antilogism* (cf. Ladd-Franklin [24]).

Consider, for instance, the valid syllogism,

All men are mortal
All Greeks are men
Therefore, all Greeks are mortal.

According to Ladd-Franklin [24], from any valid syllogism we may form a *triad*, the set that contains the premises and the negation of the conclusion (which will compose an antilogism); in this case, the corresponding antilogistic triad is:

All men are mortal
All Greeks are men
Not all Greeks are mortal.

The triad clearly forms an inconsistent set, but instead of being rejected, an inconsistent triad gives rise to three different valid syllogisms. In this case, besides the one used to generate the triad, we also have:

All Greeks are men
Not all Greeks are mortal
Therefore, not all men are mortal

and

All men are mortal
Not all Greeks are mortal
Therefore, not all Greeks are men

Emil Post, who received his doctorate from Columbia in 1920 for a dissertation proving the consistency of the propositional calculus of Whitehead and Russell's *Principia Mathematica*, had a purely syntactical notion of consistency: a logical system is *Post-consistent* if no propositional variable alone is a theorem of the system.

On the other hand, the notion of consistency of D. Hilbert (sometimes called *absolute consistency*) coincides with what we call *non-triviality*: there must be at least sentence which is not probable in the system.

But there is also the notion of λ-*non-triviality*: a set of sentences is λ-non-trivial w.r.t. a sentence λ, if it does not derive just λ. Of course, such notions may coincide in some logics (they do coincide in classical propositional logic, for instance), but are not necessarily coextensive.

Against blind acceptability of such notions of consistency there are at least two arguments: first, they only coincide in classical reasoning. If logic is more than "classical", we have to face that many other senses of consistency may be lurking.

Second, it is so easy to justify the *frisson* and the energy spent on exorcising contradictions *a priori*. The "later" Wittgenstein denied the need of a proof of consistency (in the sense of contradiction-free): a contradiction should not destroy the calculus.

> If a contradiction were now actually found in arithmetic — that would only prove that an arithmetic with such a contradiction in it could render very good service. Wittgenstein ([49], 7:35).
>
> [The idea that] a contradiction destroys the calculus can with a little imagination certainly be shaken (Wittgenstein [49], 7:15).

Not only notions of consistency are not coincident outside classical logic, but it is also interesting to recall that "maximality" with respect to such notions (*maximal non-triviality* and *maximal* λ-*non-triviality*) is what obtains semantical completeness in several non-classical logics. Thus neither completeness nor consistency are absolute.

Other areas may have still distinct notions of consistency: consistency is seen as the absence of locks in a database system, and some people in quantum physics talk about consistency as self-decoherence. Some selective definitions of consistency have already been given, as in Robles and Méndez [38], where consistency is understood as the absence of the negation of a theorem, and not as the absence of any contradiction. Although such *ad hoc* definitions have their own interest, they are not totally satisfying.

As pointed out in Woleński [50], truth and consistency cannot be equated, as highlighted by Russell in [40] more than a century ago; truth implies consistency, but not vice-versa: one might, so, formulate a consistent story which could be notoriously false. This objection to the coherence theory of truth supports the intuition that coherence should be thought as something stronger than mere consistency.

The asymmetry between truth and consistency, the surplus of senses of consistency, and the divergent notions of consistency even in arithmetical theories discussed in Section 3 point to the prospect of treating consistency from a loftier level.

6. Consistency taken structuralistically?

The acid debate on the foundations of geometry between Russell and Poincaré in the crepuscule of the nineteenth century ended with Hilbert's *Grundlagen der Geometrie* of 1899, when a new era of mathematics arose. More than a mechanist formalism, the axiomatic approach can be viewed as connected to the idea of structuralism (as defended in Shapiro [43]), the notion that what matters to a concept is how it relates to others.

Even if we take for granted the notion of completeness (in the sense that a set of axioms is complete if, for any statement in the language, either that statement or its negation is provable from the axioms) as commonly accepted, we should be aware that there are other notions of completeness as e.g. the sense of *Post-completeness*, by which addition of any unprovable sentence as an axiom results in a trivial system. Although some logical systems, as **PC**, are Post-complete, predicate logic and most modal logics are not. As much as completeness is not absolute, other approaches to consistency may be influent on understanding Gödel's Second Incompleteness Theorem.

Gödel's results (later refined by Rosser) originally presupposed the notion of ω-*consistency*, where a theory T is ω-inconsistent if it is possible to prove a sort of universal property about numbers (that is, T proves $P(n)$ for every standard natural number n), and, at the same time, it is possible to prove that there is some (non-standard) number $\mathfrak{m}$ such that $P(\mathfrak{m})$ fails.

An ω-consistent theory is not only (syntactically) non-contradictory (that is, does not prove any single contradiction), but also avoids proving certain infinite collections of sentences that are intuitively contradictory. Reciprocally, the idea of ω-inconsistency is a weak form of inconsistency, and may come to mind as a rational defense in certain social situations, as in the following nice example:

> A little girl of four years of age was making, at her dinner, the interesting example of eating her soup with a fork. Her nurse said to her, "Nobody eats soup with a fork, Emily," and Emily replied, "But I do, and I am somebody". (Ladd-Franklin [24], p. 532).

Boolos has shown in [5] that the concept of ω-consistency (which is clearly distinct from non-contradictoriness) can be treated in modal terms: indeed, the well-known system **KGL** is the modal logic of ω-consistency. And Boolos has also shown (a specially nice proof appears in Boolos [6]) how Gödel's Second Incompleteness Theorem can be easily proven in modal logic assuming the modal version of Hilbert-Bernays-Löb provability conditions, and taking consistency as $\neg\Box\bot$, where $\bot$ is a contradiction and $\Box$ is the modal translation of the sentence $\exists x Proof(x, \ulcorner p \urcorner)$ (whose meaning is that x codifies the proof of the sentence whose Gödel number is $\ulcorner p \urcorner$, or simply that p is provable in the theory).

It is essential in Boolos argument, however, that provability of p ($\Box p$) is consistent for any p, very much in the sense of the logics of formal inconsistency (the LFI's as in Carnielli, Coniglio and Marcos [15]). This gives us hope to prove (or at least the daring to conjecture) that a variant (indeed, a generalization) of Gödel's Second Incompleteness Theorem may be viable, in the following form:

> If an adequate theory T is non-trivial, then the sentence in the language of T stating that T is consistent is not provable in T; and moreover, *this* statement is provable in T.

In a book with a distinctive approach to the nature of mathematical ideas, Byers in [9] claims that creativity and understanding (specially in mathematics) arise out of problematic situations, and that there are several significant results in mathematics that are only accessible to reasoning by contradiction. Examples include randomness, chaos and complexity theory, complex numbers, Gödelian statements (statements that are true, but not derivable from a given formal system, as the well-known Goodstein's theorem). Moreover, several founding concepts in mathematics are indeed essentially contradictory, as the ideas of zero and infinity.

Byers sees "naive realism" (the presupposition of a one-to-one correspondence between objects of thought and objects of the natural world), a form of it being Platonism in mathematics, as fundamentally opposed to the mathematical reasoning governed by non-contradiction and self-reference.

The book defends the remarkable position that the role assigned to logic and consistency within mathematical activity should be revaluated, since mathematics is quintessential human activity, and human beings are capable of simultaneously sustaining two contradictory points of view.

However, the author inadvertently mistakes consistency with non-contradiction, a distinction which would make his discussion much more compelling. Worse than this, he fails to take into account that this same revaluation can be of advantage to logic itself:

> Logic and rationality imply consistency, and consistency means the avoidance of contradiction. (Byers [9], p. 82).

and

> Logic abhors the ambiguous, the paradoxical, and especially the contradictory, but the creative mathematician welcomes such problematic situations because they raise the question, 'What is going on here?'. (Byers [9], p. 6).

Not only logic *does not* necessarily abhor the ambiguous and the contradictory, but LFI's (paraconsistent logic), as we have discussed, offer a judicious tool to reason with contradictory or problematic situations.

However, it is positively surprising that a book on mathematical creativity makes a concession on the local versus global character of consistency, a positions which is perfectly in-line with the intuition of possible-translations semantics (PTS's) discussed in Section 2:

> Nevertheless, it is possible to take another point of view, one in which noncontradiction is not absolute. In this view consistency would be a local not a global phenomenon. (Byers [9], p. 83).

Consistency, in the sense of the LFI's, is a primitive notion, a concept guided by its own set of axioms, and there is nothing inherently insurmountable in starting from an elaborate notion of consistency, which would not only completely modify the sense of philosophical barrier that Gödel's Second Incompleteness Theorem represents, but to be of extraordinary benefit to the epistemology of mathematics, and consequently to science as a whole. The sketch, at least, is now ready.

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